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Uniformly valid polynomial representations for boundary-layer problems

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Abstract This paper is concerned with finding uniformly valid polynomial solutions to two-point boundary-layer problems for ordinary differential equations using two-point Hermite expansions. First it is shown that a direct method, generalised to include the fitting of different orders of derivatives at the end points of the interval, fails as the small parameter in the problem becomes small. A novel two-scale approach is used to obviate this failure which is caused by the singular behaviour of the solution and its derivatives in the boundary layer. The method is introduced and illustrated by a series of examples. It is believed that with the use of a symbolic computational facility such as MAPLE the technique is a viable and competitive alternative to the classical methods of matched expansions and multiple scales.

Keywords Asymptotics · Boundary layers · Boundary-value problems · Hermite interpolation · Singular perturbations

1 Introduction

In recent years the author has applied global Hermite interpolation to a variety of boundary-value problems for ordinary and partial differential equations ([1–4]) and earlier to ordinary differential equations [5]. The aim of the present paper is to use these ideas to construct uniformly valid polynomial solutions to ordinary boundary-value problems containing a small parameter and which exhibit a boundary layer. In particular, we consider model problems of the form

$$\varepsilon y'' + a(x, \varepsilon)y' + b(y, x, \varepsilon) = 0, \quad y(0) = \alpha, \quad y(1) = \beta, \quad \varepsilon > 0 \tag{1.1}$$

in which ε is small and where $a(x, \varepsilon) > 0$ on [0,1] with the boundary layer located at an end point. We present the method as a competitive and viable alternative to the classical methods of matched expansion and multiple scales for such problems. Problems of this type occur in almost every field of applied mathematics, be it fluid dynamics, biology or science and engineering in general and over the years a number of techniques have been devised to solve them. The literature on both methodology and applications is extensive wherein the classic books by Van Dyke

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[6, Chap. V] and Nayfeh [7, Chaps. 4 and 6] have been superseded and extended by a number of texts such as Holmes [8, Chaps. 2 and 3] to which the reader is referred for background material on techniques, applications and historical perspective.

The outline of the paper is as follows. In Sect. 2 we explain what we mean by Hermite expansions and give algorithms for their construction together with a discussion of the convergence theory in the complex plane. In Sect. 3 a direct method using Hermite expansions, introduced by Grundy [1], is shown to fail as ε is reduced toward zero. However, we are able to get round the difficulty using a two-scale approach which enables the limit $\varepsilon \to 0$ to be taken. In Sect. 4 we develop this method via a series of examples which are intended to illustrate the accuracy and applicability of the technique. We compare our results with numerical solutions and also with the classical method of multiple scales [7, Chap. 6]. Throughout the paper MAPLE is used as an indispensable facilitator of the algebraic manipulations and also as a rootfinder and BVP/IVP solver.

2 Two-point Hermite expansions

A Hermite expansion [9] of a function $f(x)$ can be regarded as a generalisation of a Taylor expansion. If we look upon the Taylor expansion as fitting derivatives of the function at a single given point, then the Hermite expansion performs the same role at any number of given points. In this paper we are particularly interested in constructing expansions which fit derivatives at two such points. These expansions are sometimes referred to as two-point Taylor expansions.

The first question we address is how do we construct the two-point analogue of the Taylor polynomials, namely polynomials whose derivatives up to prescribed orders fit the corresponding derivatives of a given function $f(x)$ at two points which, without loss of generality, we may take to be $x = 0$ and $x = 1$. In general, the orders of derivatives fitted may be different, so we are seeking a polynomial $H_N(x)$ of degree N such that

$$H_N^{(k)}(0) = f^{(k)}(0), \quad k = 0, 1, \ldots\ldots, n_0 - 1 \tag{2.1}$$

and

$$H_N^{(k)}(1) = f^{(k)}(1), \quad k = 0, 1, \ldots\ldots, n_1 - 1. \tag{2.2}$$

where $N = n_0 + n_1 - 1$. The fact that such a polynomial exists and is unique is well known (See for example Davis [10, Chaps. 3 and 4]) as is an algorithm for its construction (Stoer and Bulirsch [11, pp. 52–54]). This algorithm can be written in the form

$$H_N(x) = \sum_{j=0}^{n_0-1} L_{0j}(x) f^{(j)}(0) + \sum_{j=0}^{n_1-1} L_{1j}(x) f^{(j)}(1). \tag{2.3}$$

where the L_{ij} are given by

$$L_{0,n_0-1} = l_{0,n_0-1}, \quad L_{1,n_1-1} = l_{1,n_1-1}$$

and for $k_0 = n_0 - 2, n_0 - 3, \ldots\ldots\ldots, 0$,

$$L_{0k_0} = l_{0k_0} - \sum_{v=k_0-1}^{n_0-1} l_{0k_0}^{(v)}(0) L_{0v}$$

while for $k_1 = n_1 - 2, n_1 - 3, \ldots\ldots\ldots, 0$,

$$L_{1k_1} = l_{1k_1} - \sum_{v=k_1+1}^{n_1-1} l_{1k_1}^{(v)}(1) L_{1v}$$

In the above

$$l_{0j} = \frac{x^j (1-x)^{n_1}}{j!}, \quad 0 \le j \le n_0$$

and

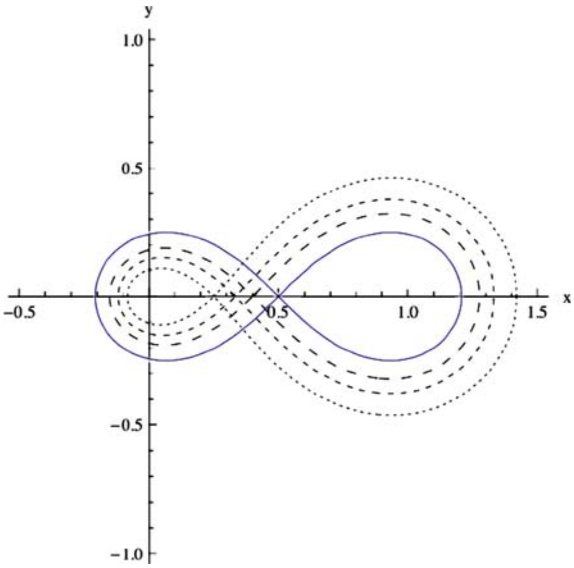

Fig. 1 The critical ovals O_μ for ______ $\mu = 1$, __ __ __ $\mu = 2/3$, _ _ _ _ $\mu = 1/2$, ... $\mu = 1/3$

$$l_{1j} = \frac{(x-1)^j x^{n_0}}{j!}, \quad 0 \le j \le n_1.$$

The representation (2.3), which is systematically programmable, is called the two-point Hermite interpolating polynomial of $f(x)$. We note for future reference that (2.3) can be written directly in terms of the Taylor coefficients of $f(x)$ at $x = 0$ and $x = 1$ via their relationship with the derivatives.

If we regard (2.3) as an expansion of $f(x)$, then one can construct an expression for the remainder r_N in the form

$$r_N = f(x) - H_N(x) = \frac{f^{(N+1)}(\xi) x^{n_0} (x-1)^{n_1}}{(N+1)!}, \tag{2.4}$$

which is analogous to the Taylor-expansion remainder. It turns out that an associated convergence theory can be developed for Hermite expansions in the complex plane in the limit $n_0 \to \infty$ and $n_1 \to \infty$. This theory is well known when $n_0 = n_1$ and the number of terms fitted at each end point is the same, (See for example the book by Davis [10] or more recently the paper of Lopez and Temme [12]). However, what appears to be less appreciated is that it is possible, by fitting different numbers of terms at the end points, to consider more general expansions in which n_0 and n_1 approach infinity at different rates. Let us suppose that we take the limit $n \to \infty$ such that $n_0 = np$ and $n_1 = nq$ where p and q are prescribed integers such that $\mu = p/q$ is in its lowest terms. When $\mu = 1$, it is known that the domain of convergence of the corresponding Hermite expansion is the interior of the maximal Cassini oval, with foci at $z = 0$ and $z = 1$ and symmetric about $z = 1/2$, within which $f(z)$ is analytic. Significantly, however, for $\mu \ne 1$ these become generalised Cassini ovals and are no longer symmetric about $z = 1/2$; hence the domains of convergence are different. If these ovals contain the interval [0,1] on the real axis, the relevant Hermite expansion will certainly converge on that interval. Thus we identify in Fig. 1, for various values of μ, the critical ovals O_μ which just contain the interval [0,1]. If $f(z)$ has singularities inside an O_μ, the corresponding Hermite expansion will fail to converge throughout the interval [0,1], since in that event the Cassini ovals have disjoint branches and do not include the whole of [0,1].

For future reference we note that the Hermite expansion of an entire function will converge everywhere. Since the domains of convergence can be modified by allowing $\mu \ne 1$, this gives us more flexibility regarding the maintenance of convergence when singularities are present or, as we shall see later, the acceleration of convergence for entire functions. Further details of the convergence theory when $\mu \ne 1$ are given in [13], but we give an outline of the main convergence result in Appendix 1.

3 Application to the solution of second-order boundary-value problems

We consider the boundary-value problem (1.1), namely

$$\varepsilon y'' + a(x,\varepsilon)y' + b(y,x,\varepsilon) = 0 \tag{3.1}$$

$$y(0) = \alpha, \quad y(1) = \beta, \tag{3.2}$$

where $\varepsilon > 0$.

A direct application of Hermite interpolation involves replacing $y(x)$, or an equivalent formulation of it, by an $H_N(x)$ of the form (2.3); see [1]. The first step is to write down the Taylor series for $y(x)$ about $x = 0$ and $x = 1$. From (3.1) these are

$$y(x) = \alpha + a_1(\varepsilon)x + \sum_{i=2}^{\infty} a_i\,(a_1(\varepsilon),\varepsilon)\,x^i \tag{3.3}$$

and

$$y(x) = \beta + (x-1)b_1(\varepsilon) + \sum_{i=2}^{\infty} b_i(b_1(\varepsilon),\varepsilon)(x-1)^i, \tag{3.4}$$

in which the coefficients $a_i(\varepsilon)$ and $b_i(\varepsilon)$, $i \geq 2$, depend on the two unknown coefficients $a_1(\varepsilon)$ and $b_1(\varepsilon)$ as indicated. We now construct an $H_N(x)$ from (3.3) and (3.4) of the form (2.3) and use it as a replacement in the problem defined by (3.1) and (3.2).

We have found that, in general, accuracy is improved by recasting (3.1) and (3.2) into an integral formulation rather than using (3.1) itself, so we start off by integrating (3.1) to obtain

$$\varepsilon y'(x) + a(x,\varepsilon)y(x) - \varepsilon a_1 - a(0,\varepsilon)\alpha + \int_0^x \left\{ b(y(s),s,\varepsilon) - y(s)\frac{\mathrm{d}a(s,\varepsilon)}{\mathrm{d}s} \right\} \mathrm{d}s = 0 \tag{3.5}$$

and again to get

$$\varepsilon y(x) - \varepsilon\alpha - x\varepsilon a_1 - a(0,\varepsilon)\alpha x + \int_0^x a(s,\varepsilon)y(s)\mathrm{d}s + \int_0^x (x-s)\left\{ b(y(s),s,\varepsilon) - y(s)\frac{\mathrm{d}a(s,\varepsilon)}{\mathrm{d}s} \right\} \mathrm{d}s = 0. \tag{3.6}$$

Putting $x = 1$ in (3.5) and (3.6) gives

$$\varepsilon(b_1 - a_1) + a(1,\varepsilon)\beta - a(0,\varepsilon)\alpha + \int_0^1 \left\{ b(y(s),s,\varepsilon) - y(s)\frac{\mathrm{d}a(s,\varepsilon)}{\mathrm{d}s} \right\} \mathrm{d}s = 0 \tag{3.7}$$

and

$$\varepsilon(\beta - \alpha) - \varepsilon a_1 - a(0,\varepsilon)\alpha + \int_0^x (x-s)\left\{ b(y(s),s,\varepsilon) - y(s)\frac{\mathrm{d}a(s,\varepsilon)}{\mathrm{d}s} \right\} \mathrm{d}s = 0. \tag{3.8}$$

We now replace $y(s)$ in (3.7) and (3.8) by an $H_N(x)$ which generates the required two equations for the unknown pair $\{a_1(\varepsilon), b_1(\varepsilon)\}$. Once these are found we may construct the desired Hermite interpolant from (3.7) and complete what we refer to as the direct method.

Before going any further, we make some comments regarding the general applicability of the direct method in the light of the convergence theory for Hermite expansions alluded to in Sect. 2 and also how it compares with other methods for constructing polynomial solutions to boundary-value problems. For linear equations we can immediately locate the singularities of the solution in the complex plane from the equation itself. If any of these singularities lie within a critical oval, the corresponding Hermite expansion will not converge on the whole of the interval [0,1] and the method will fail, although we do have an inbuilt flexibility since we can modify the convergence domain via the choice of μ. For equations with no singular points in the finite complex plane the solutions will be entire functions and there will be no problem regarding convergence, although the rate of convergence may be an issue. For nonlinear problems the situation is less clear cut since we do not know in advance the position of any singularities. We will say more about this when we consider a nonlinear example in Sect. 4.3. A detailed discussion of how singularities may affect the implementation of the direct method is given in [13].

The convergence difficulties when singularities are present is an important consideration when we consider comparable methods for the construction of solutions of boundary-value problems. These alternatives usually involve collocation of Chebychev or Legendre expansions at the zeros of the respective orthogonal polynomials. Provided the solution is not singular on [0,1] itself, these methods will always converge irrespective of the location of complex singularities although rates of convergence can be adversely affected if singularities encroach too close. However, we would like to point out that these methods have a distinct drawback when compared with methods involving Hermite expansions. For Hermite expansions the number of unknowns is fixed, being independent of the order of the expansion, whereas in collocation at selected points the number of unknowns increases with the order of the expansion. This is of particular importance for rootfinding involved in solving nonlinear problems.

In the linear examples which follow we are specifically dealing with boundary-layer problems in which the solutions are entire functions, so we have no difficulty with complex singularities. However, even though the solutions to such problems converge everywhere in the complex plane, rates of convergence become problematical due to the essential singularity involving $\mathrm{e}^{-x/\varepsilon}$ as $\varepsilon \to 0$. For the nonlinear example there is a singularity outside the interval [0,1] but it can be excluded by a suitable choice of μ. As we shall see, a judicious choice of μ can significantly improve rates of convergence for both linear and nonlinear problems.

To illustrate how we implement the above direct method for problems like (1.1), we consider a specific example

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1$$

for which $a(x, \varepsilon) = 1 + \varepsilon$, $b(y, x, \varepsilon) = 1$, $\alpha = 0$ and $\beta = 1$. This problem has the exact solution

$$y(x) = \frac{\mathrm{e}^{-x} - \mathrm{e}^{-x/\varepsilon}}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}. \tag{3.9}$$

We present the results for the direct method in the upper half of Table 1 for various values of ε.

Note that here and throughout the paper we round to five significant figures and, in order to give a visual impression of the convergence with n, *bold* digits represent ‘unconverged digits’. We observe that we can take $\mu = p/q = 1$ down to $\varepsilon = 0.1$ and retain acceptable convergence. The essential singularity at infinity then becomes too severe. However, we can maintain a practical rate of convergence and reduce ε to 0.01 by taking $\mu = 1/3$ and weighting $x = 1$, being the end point closest to the essential singularity. Nevertheless, the direct method will ultimately fail as a practical option for sufficiently small ε. The purpose of this paper is to show how we can obviate this failure and adapt the direct method to construct solutions for values of ε as small as we please.

The difficulty clearly arises from the nonuniformity in the boundary layer, located where $x = O(\varepsilon)$, wherein derivatives of all orders become large thereby increasing the error without bound as $\varepsilon \to 0$. To deal with this we introduce a second independent variable

$$z = \frac{x}{\varepsilon} \tag{3.10}$$

and regard the independent variable as a function of x and z. We then follow the usual multiple scales philosophy and write

$$y(x) \equiv Y(x, z)$$

recasting (1.1) as

$$\frac{\partial^2 Y}{\partial z^2} + (\varepsilon+1)\frac{\partial Y}{\partial z} + \varepsilon^2\frac{\partial^2 Y}{\partial x^2} + \varepsilon(\varepsilon+1)\frac{\partial Y}{\partial x} + \varepsilon Y + 2\varepsilon\frac{\partial^2 Y}{\partial x \partial z} = 0. \tag{3.11}$$

The novelty now is to expand

$$Y(x, z) = \sum_{i=0}^{\infty} A_i(z, \varepsilon) x^i \tag{3.12}$$

Table 1 Results for $\varepsilon y'' + (\varepsilon + 1)y' + y = 0,\ y(0) = 0,\ y(1) = 1$ as ε is reduced

ε	μ	n	$y'_A(0)$	$y'_A(1)$	$y_A(0.05)$	$y_A(0.5)$	Method
0.5	1	4	4.3003	−.41**799**	.19950	1.0263	Direct
		6	4.3003	−.41802	.19950	1.0263	
		8	4.3003	−.41802	.19950	1.0263	
		Exact	4.3003	−.41802	.19950	1.0263	
0.1	1	8	24.46**8**	−.998**77**	.9371**1**	1.630**7**	
		10	24.468	−.99889	.93710	1.6306	
		12	24.468	−.99889	.93710	1.6306	
		Exact	24.468	−.99889	.93710	1.6306	
0.01	1/3	16	269.**06**	−1.0	1.56**69**	1.648**6**	
		18	269.11	−1.0	2.5674	1.6487	
		20	269.11	−1.0	2.5674	1.6487	
		Exact	269.11	−1.0	2.5674	1.6487	
0.01	1	2	269.11	−1.0	2.5674	1.6487	Two-scale
		3	269.11	−1.0	2.5674	1.6487	
0.001	1	2	2715.6	−1.0	2.5857	1.6487	
		3	2715.6	−1.0	2.5857	1.6487	
		Exact	2715.6	−1.0	2.5857	1.6487	

In this and the following tables the direct method fits np and nq terms of the Taylor series at $x = 0$ and $x = 1$, respectively, where $\mu = p/q$ in its lowest terms. The degree of the resulting Hermite expansion is given by $N = n(p + q) - 1$. The two-scale method fits $n + 1$ terms of each of the series (3.12) and (3.13): the degree of the resulting polynomial is $2n + 1$ in x

about $x = 0$ in which the series coefficients are functions of z. The series about $x = 1$, however, need not contain a z-dependence, so we write as in (3.4)

$$Y(x, z) = 1 + B_1(\varepsilon)(x - 1) + \sum_{i=2}^{\infty} B_i(B_1(\varepsilon), \varepsilon)(x - 1)^i \tag{3.13}$$

in which $B_1(\varepsilon)$ is the only unknown. We now substitute (3.12) in (3.11) to give a hierarchy of equations, $\{D_n\}$ $n = 0, 1, 2, \ldots,$ for the $A_i(x, z)$ of the form

$$D_0 : \{2A_2 + A_1\}\varepsilon^2 + \left\{A'_0 + 2A'_1 + A_0 + A_1\right\}\varepsilon + A''_0 + A'_0 = 0, \tag{3.14}$$

$$D_1 : \{6A_3 + 2A_2\}\varepsilon^2 + \left\{A'_1 + 4A'_2 + A_1 + 2A_2\right\}\varepsilon + A''_1 + A'_1 = 0, \tag{3.15}$$

$$D_2 : \{12A_4 + 3A_3\}\varepsilon^2 + \left\{A'_2 + 6A'_3 + A_2 + 3A_3\right\}\varepsilon + A''_2 + A'_2 = 0, \tag{3.16}$$

etc.
where primes denote derivatives with respect to z.

The idea now is to expand the A_i in powers of ε in such a way that secular terms on the z scale in such a series are eliminated in the limit $z \to \infty$. Thus we write

$$A_i(z, \varepsilon) = A_{i0}(z, \varepsilon) + \varepsilon A_{i1}(z, \varepsilon) + \varepsilon^2 A_{i2}(z, \varepsilon) + \cdots. \tag{3.17}$$

To illustrate how we compute the $A_{ij}(z, \varepsilon)$ let us suppose we wish to construct an $H_5(x, z)$ with $n_0 = n_1 = 2$ from (3.12) and (3.13). To do this we need to obtain the coefficients $A_i(z, \varepsilon),\ i = 0, 1, 2$. Now with (3.17) we can only obtain closed systems for these coefficients if we use (3.14) to $O(\varepsilon^2)$, (3.15) to $O(\varepsilon)$ and (3.16) to $O(1)$. This implies that in such a scheme we can only derive the truncated forms

$$A_0(z, \varepsilon) = A_{00}(z, \varepsilon) + \varepsilon A_{01}(z, \varepsilon) + \varepsilon^2 A_{02}(z, \varepsilon), \tag{3.18}$$

$$A_1(z,\varepsilon) = A_{10}(z,\varepsilon) + \varepsilon A_{11}(z,\varepsilon), \tag{3.19}$$

$$A_2(z,\varepsilon) = A_{20}(z,\varepsilon). \tag{3.20}$$

With $\alpha_{ij}(\varepsilon)$ and $\beta_{ij}(\varepsilon)$ arbitrary constants of integration in what follows, we now work sequentially to compute $A_0(z,\varepsilon)$, $A_1(z,\varepsilon)$ and $A_2(z,\varepsilon)$. The sequence of operations in what follows are indicated in bold type.

$$\begin{aligned} &\textbf{(3.14) to } \boldsymbol{O(1)}\text{:} && A''_{00} + A'_0 = 0, \\ &\quad\text{giving} && A_{00} = \alpha_{00} + \beta_{00}\mathrm{e}^{-z}. \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\textbf{(3.15) to } \boldsymbol{O(1)}\text{:} && A''_{10} + A'_{10} = 0, \\ &\quad\text{giving} && A_{10} = \alpha_{10} + \beta_{10}\mathrm{e}^{-z}. \end{aligned} \tag{3.22}$$

(3.16) to $O(\varepsilon)$: $\quad A''_{01} + A'_{01} = -A'_{00} - 2A'_{10} - A_{00} - A_{10} = (\alpha_{00} + \alpha_{10}) + \beta_{10}\mathrm{e}^{-z}$,

using (3.21) and (3.22). Eliminating secular terms from A_{01} requires

$$\alpha_{10} = -\alpha_{00}, \quad \beta_{10} = 0 \tag{3.23}$$

and so we can write

$$A_{01} = \alpha_{01} + \beta_{01}\mathrm{e}^{-z} \tag{3.24}$$

$$\begin{aligned} &\textbf{(3.14) to } \boldsymbol{O(1)}\text{:} && A''_{20} + A'_{20} = 0, \\ &\quad\text{giving} && A_{20} = \alpha_{20} + \beta_{20}\mathrm{e}^{-z}. \end{aligned} \tag{3.25}$$

(3.13) to $O(\varepsilon)$: $\quad A''_{11} + A'_{11} = -4A'_{20} - A'_{10} - 2A_{20} - A_{10} - (\alpha_{10} + 2\alpha_{20}) + 2\beta_{20}\mathrm{e}^{-z}$,

using (3.22), (3.23) and (3.25). Eliminating secular terms in A_{11} then requires

$$\alpha_{20} = -\frac{\alpha_{10}}{2} = \frac{\alpha_{00}}{2}, \quad \beta_{20} = 0 \tag{3.26}$$

and hence

$$A_{11} = \alpha_{11} + \beta_{11}\mathrm{e}^{-z} \tag{3.27}$$

$$\begin{aligned} \textbf{(3.14) to } \boldsymbol{O(\varepsilon^2)}: \quad A''_{02} + A'_{02} &= -A'_{01} - 2A'_{11} - A_{01} - A_{11} - 2A_{20} - A_{10} \\ &= -(2\alpha_{20} + \alpha_{10} + \alpha_{01} + \alpha_{11}) - (2\beta_{20} + \beta_{10} + \beta_{11})\,\mathrm{e}^{-z} \\ &= -(\alpha_{01} + \alpha_{11}) - \beta_{11}\mathrm{e}^{-z}, \end{aligned}$$

using (3.24–3.27). Eliminating secular terms requires

$$\alpha_{11} = -\alpha_{01}, \quad \beta_{11} = 0 \tag{3.28}$$

so that

$$A_{02} = \alpha_{021} + \beta_{02}\mathrm{e}^{-z}, \tag{3.29}$$

which completes the secularity elimination procedure.

We now observe that $A_0(z)$ can be written as

$$A_0(z) = \sum_{i=0}^{2} \left(\alpha_{i0} + \beta_{i0}\mathrm{e}^{-z}\right)\varepsilon^i,$$

so the sum can be telescoped into a single leading term

$$A_0(z) = \alpha_{00}(\varepsilon) + \beta_{00}(\varepsilon)\mathrm{e}^{-z}, \tag{3.30}$$

effectively putting

$\alpha_{0i} = \beta_{0i} = 0, \quad i = 0, 1, 2$

for all i in what has gone before. Furthermore from (3.30) the boundary condition at $x = z = 0$ gives

$$\beta_{00} = -\alpha_{00}, \tag{3.31}$$

so that the secularity conditions (3.23), (3.26), (3.28) become

$$\alpha_{10} = -\alpha_{00}, \quad \alpha_{20} = \frac{\alpha_{00}}{2}, \quad \alpha_{11} = 0; \quad \beta_{10} = \beta_{20} = \beta_{11} = 0.$$

Thus, all the coefficients in (3.18–3.20) and hence in the series (3.12) depend on the single unknown $\alpha_{00}(\varepsilon)$. Recalling that the series (3.13) about $x = 1$ contains only the single unknown $B_1(\varepsilon)$, we observe that the resulting two-point expansion $H_5(x, z; \varepsilon)$ depends solely on the unknown pair $\{\alpha_{00}(\varepsilon), B_1(\varepsilon)\}$ and can be written in the form

$$\begin{aligned} H_5(x, z) = {} & \alpha_{00}(1 - \mathrm{e}^{-z}) - \alpha_{00}x + \frac{1}{2}\alpha_{00}x^2 + x^3 \left\{ \frac{1 + B_1}{2\varepsilon} + 9B_1 - 20 + \alpha_{00}(11 - 20\mathrm{e}^{-z}) \right\} \\ & + x^4 \left\{ \frac{1 + B_1}{\varepsilon} + 16B_1 - 30 + \alpha_{00}(17 - 30\mathrm{e}^{-z}) \right\} \\ & - x^5 \left\{ \frac{1 + B_1}{2\varepsilon} + 7B_1 - 12 + \alpha_{00}(7 - 12\mathrm{e}^{-z}) \right\}. \end{aligned} \tag{3.32}$$

With the help of a computational facility such as MAPLE the above scheme can be used to construct any $H_N(x)$ with $n_0 = n_1 = n$. Thus, with

$$A_i(z, \varepsilon) = \sum_{k=0}^{n-i} A_{ik}\varepsilon^k \tag{3.33}$$

and

$$A_{ik} = \alpha_{ik} + \beta_{ik}\mathrm{e}^{-z} \tag{3.34}$$

the systematically programmable scheme for substituting the $\{A_n(z, \varepsilon)\}$ in the $\{D_n\}$ for $n = 0, 1, 2, 3, \ldots$ and computing all the α_{ij} and β_{ij} in terms of α_{00} and β_{00} by eliminating secular terms is presented in Table 2.

Since the boundary condition at $x = 0$ gives β_{00} in terms of α_{00}, it only remains to compute the unknown pair $\{\alpha_{00}(\varepsilon), B_1(\varepsilon)\}$.

We do this by following the procedure used in the direct method but now starting from (3.11). Integrating with respect to the slow variable from 0 to x and applying the boundary conditions at $x = 0$ we have

$$\varepsilon^2\frac{\partial Y}{\partial x} - \varepsilon^2 A_1 + \varepsilon(\varepsilon + 1)Y - A_0\varepsilon(1 + \varepsilon) + 2\varepsilon\frac{\partial Y}{\partial z} - 2\varepsilon A_0' + \int_0^x \left\{ \frac{\partial^2 Y}{\partial z^2} + (1 + \varepsilon)\frac{\partial Y}{\partial z} + \varepsilon Y \right\} \mathrm{d}s = 0. \tag{3.35}$$

Table 2 Programmable sequential procedure for the elimination of secular terms which generate the A_{ij} and B_{ij} in terms of α_{00} and β_{00}

	$O(1)$	$O(\varepsilon)$	$O(\varepsilon^2)$	.....	$O(\varepsilon^n)$
D_0	1	3	6	…	$\frac{n(n+3)}{2} + 1$
D_1	2	5	....	$\frac{n(n+3)}{2}$	
D_2	4	....	. . .		
. .	....	$\frac{n(n+1)}{2} + 2$			
D_n	$\frac{n(n+1)}{2} + 1$				

The tabulated numbers indicate the order in which powers of ε are equated in the D_n

A second integration with respect to x gives

$$\varepsilon^2 Y - \varepsilon^2 A_0 - \varepsilon^2 A_1 x + \varepsilon(\varepsilon+1)A_0 x - 2\varepsilon A_0' x \\ + \int_0^x \left\{ (x-s)\left(\frac{\partial^2 Y}{\partial z^2} + (\varepsilon+1)\frac{\partial Y}{\partial z} + \varepsilon Y\right) + \varepsilon(\varepsilon+1)Y + 2\varepsilon \frac{\partial Y}{\partial z} \right\} \mathrm{d}s = 0, \tag{3.36}$$

where primes denote ordinary derivatives with respect to z. We now apply the boundary conditions at $x = 1$ by putting $x = 1, z = 1/\varepsilon$ in the above integral forms. On replacing $Y(\cdot, z)$ by $H_N(\cdot, z)$ we obtain two equations for the two unknowns pair $\{\alpha_{00}(\varepsilon), B_1(\varepsilon)\}$. The solution of which enables us to complete the construction of the required polynomial.

As a check on the above calculation and to put things into perspective, we compute the various coefficients for the above example from the exact solution (3.9). Here

$$Y(x, z) = \frac{\mathrm{e}^{-x} - \mathrm{e}^{-z}}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}$$

Expanding this about $x = 0$, according to (3.12) we have

$$A_0(z, \varepsilon) = \frac{1 - \mathrm{e}^{-z}}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}, \quad A_1(z, \varepsilon) = -\frac{1}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}, \quad A_2(z, \varepsilon) = \frac{1}{2(\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon})},$$

so that

$$A_{00}(z, \varepsilon) = \frac{1 - \mathrm{e}^{-z}}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}, \quad A_{01}(z, \varepsilon) = A_{02}(z, \varepsilon) = 0,$$

$$A_{10}(z, \varepsilon) = -\frac{1}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}, \quad A_{11}(z, \varepsilon) = 0, \quad A_{20}(z, \varepsilon) = \frac{1}{2(\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon})},$$

which checks with our analysis giving

$$\alpha_{00}(\varepsilon) = \frac{1}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}}.$$

Finally, expanding the exact solution (3.9) about $x = 1$, we have

$$B_1(\varepsilon) = \frac{\mathrm{e}^{1-1/\varepsilon} - \varepsilon}{\varepsilon(1 - \mathrm{e}^{1-1/\varepsilon})}.$$

We present the results of our calculations using the two-scale method for this example in the lower half of Table 1. The effectiveness of the method is clearly apparent as $\varepsilon \to 0$.

This procedure can be immediately extended, with comparable results, to linear equations where $a(x, \varepsilon)$ is a constant but $b(y, x, \varepsilon)$ has an x-dependence and an example of this is given later. When $a(x, \varepsilon)$ depends on x, some modification is necessary but the method still works well. The method can also be used when $b(y, x, \varepsilon)$ has a nonlinear dependence on y, although some non-trivial modifications have to be made. We illustrate these generalisations in the examples below where we compare our method with the orthodox method of multiple scales (See Appendix 2). Since in constructing an $H_N(x, z)$ we neglect terms of $O(\varepsilon)$ in A_{n_0}, it would seem appropriate to compare with the zero-order multiple-scales result which is what we do in the tables. We discuss this further in Sect. 5. We remark that the two-scale method is more general in some respects than the classical multiple-scales method. Even for linear equations it is not always possible to explicitly solve in closed form the ordinary differential equations which arise in that method and which govern the long-scale variation. In such cases it is often extremely awkward, even with the help of a tool such as MAPLE, to extend the solution to high order in ε. The same comment goes for the method of matched expansions where we are faced with the additional problem of doing the matching which is difficult to perform in any systematically programmable way, particularly to high order.

4 Some illustrative examples

4.1 $b(y, x, \varepsilon)$ has an x-dependence

Here we consider the linear problem

$$\varepsilon y'' + y' - xy = 0, \quad y(0) = y(1) = 1 \tag{4.1}$$

where $\varepsilon > 0$. The direct method can be applied immediately and the effect of reducing ε is revealed in the upper half of Table 3 where we can reduce ε to 0.01 by taking $\mu = 1/4$.

As far as the two-scale method is concerned, we observe that there is a boundary layer at $x = 0$ of thickness $O(\varepsilon)$; consequently we write

$$y(x, \varepsilon) = Y(x, z, \varepsilon),$$

where $z = x/\varepsilon$ so that (3.1) becomes

$$\frac{\partial^2 Y}{\partial z^2} + \frac{\partial Y}{\partial z} + \varepsilon^2 \frac{\partial^2 Y}{\partial x^2} + \varepsilon \frac{\partial Y}{\partial x} - \varepsilon x Y + 2\varepsilon \frac{\partial^2 Y}{\partial x \partial z} = 0. \tag{4.2}$$

We now form the series corresponding to (3.12) and (3.13) and follow the procedure outlined in (3.14) to (3.36) in Sect. 3. In this way we may express all the A_i in terms of the single unknown $\alpha_{00}(\varepsilon)$. We then construct the integral forms by integrating (4.2) with respect to the slow variable from 0 to x and replacing $Y(\cdot, z, \varepsilon)$ with a $H_{\mathrm{N}}(\cdot, z, \varepsilon)$ to give

Table 3 Results for $\varepsilon y'' + y' - xy = 0$, $y(0) = y(1) = 1$ as ε is reduced compared with the numerical solution to the problem (Num) and zero-order multiple-scales solution (ms0)

ε	μ	n	$y'_A(0)$	$y'_A(1)$	$y_A(0.05)$	$y_A(0.5)$	Method
0.5	1	2	−.45**639**	.46**653**	.978**30**	.886**45**	Direct
		4	−.45481	.46913	.97840	.88658	
		6	−.45481	.46913	.97840	.88658	
		Num	−.45481	.46913	.97840	.88658	
0.1	1/2	4	−3.28**51**	.842**11**	.8709**1**	.735**62**	
		6	−3.2848	.84222	.87093	.73543	
		8	−3.2848	.84222	.87093	.73543	
		Num	−3.2848	.84222	.87093	.73543	
0.01	1/4	12	−38.5**63**	.98058	.6175**2**	.69262	
		14	−38.558	.98058	.61758	.69262	
		16	−38.558	.98058	.61758	.69262	
		Num	−38.558	.98058	.61758	.69262	
0.01		2	−38.55**7**	.980**77**	.6175**9**	.692**71**	Two-scale
		3	−38.55**7**	.98058	.61758	.69262	
		4	−38.558	.98058	.61758	.69262	
		Num	−38.558	.98058	.61758	.69262	
		ms0	−3**9.347**	1.**0**	.**60994**	.6**8729**	
0.001		2	−392.66	.9980**3**	.6080**8**	.687**91**	
		3	−392.66	.99801	.60807	.68783	
		Num	−392.66	.99801	.60807	.68783	
		ms0	−39**3.47**	1.**0**	.607**29**	.687**29**	

$$\varepsilon^2\left(\frac{\partial H_N}{\partial x}-A_1\right)+\varepsilon(H_N-A_0)+\int_0^x\left\{\frac{\partial^2 H_N}{\partial z^2}+\frac{\partial H_N}{\partial z}-\varepsilon s H_N\right\}\mathrm{d}s=0, \tag{4.3}$$

$$\varepsilon^2(H_N-A_0)-\varepsilon^2 x A_1-\varepsilon x A_0-2xA_0'+\int_0^x\left\{(x-s)\left(\frac{\partial^2 H_N}{\partial z^2}+\frac{\partial H_N}{\partial z}-\varepsilon s H_N\right)+2\varepsilon\frac{\partial H_N}{\partial z}\right\}\mathrm{d}s=0. \tag{4.4}$$

Applying the boundary condition at the right-hand boundary where $x=1$, $z=1/\varepsilon$ finally yields the two equations which can be solved for the unknown pair $\{\alpha_{00}(\varepsilon), B_1(\varepsilon)\}$, which effectively completes the construction of the required polynomial in powers of x with coefficients functions of z. The results for the two-scale method are given in the lower half of Table 3 where we compare with the standard zero-order multiple-scales solution given in Appendix 2 and also the numerical solution.

4.2 $a(x,\varepsilon)$ has an x-dependence

The situation when $a(x,\varepsilon)$ is not constant is somewhat more involved. This is illustrated by the following example.

$$\varepsilon y''+(1+x)y'+3y=0,\quad y(0)=y(1)=1. \tag{4.5}$$

As before, the direct method follows the familiar pattern as we attempt to reduce the value of ε and is illustrated in the upper half of Table 4. Here we can reach $\varepsilon=0.01$ by taking $\mu=1/3$.

Table 4 Results for $\varepsilon y''+(1+x)y'+3y=0,\ y(0)=y(1)=1$ as ε is reduced and comparison with the zero-order multiple-scales and numerical solution

ε	μ	n	$y_A'(0)$	$y_A'(1)$	$y_A(0.05)$	$y_A(0.5)$	Method
0.5	1	4	6.8**260**	−2.16**71**	1.316**5**	2.11**91**	Direct
		6	6.8193	−2.1687	1.3162	2.1162	
		8	6.8193	−2.1687	1.3162	2.1162	
		Num	6.8193	−2.1687	1.3162	2.1162	
0.1	1	8	67.07**1**	−1.701**6**	3.5652	2.816**1**	
		10	67.072	−1.7014	3.5652	2.8164	
		12	67.072	−1.7014	3.5652	2.8164	
		Num	67.072	−1.7014	3.5652	2.8164	
0.01	1/3	16	697.93	−1.5153	7.156**2**	2.3991	
		18	697.93	−1.5153	7.1561	2.3991	
		20	697.93	−1.5153	7.1561	2.3991	
		Num	697.93	−1.5153	7.1561	2.3991	
		ms0	6**62.00**	−1.5**000**	**6.8648**	2.3**704**	
0.01		6	697.9**5**	−1.5153	7.156**3**	2.399**0**	Two-scale
		7	697.93	−1.5153	7.1561	2.3991	
		8	697.93	−1.5153	7.1561	2.3991	
0.001		5	699**7.9**	−1.5015	6.93**97**	2.37**10**	
		6	6998.1	−1.5015	6.938**2**	2.373**2**	
		7	6998.0	−1.5015	6.9381	2.3731	
		Num	6998.0	−1.5015	6.9381	2.3731	
		ms0	69**62.0**	−1.5**000**	**6.9107**	2.37**04**	

Table 5 Direct method for nonlinear problem for $\varepsilon = 1$ with $\mu = 1$ exhibiting the two solutions to the problem for $\varepsilon > 0$, The S_1 solution has converged to five significant figures by $n = 6$ while the S_2 solution converges more slowly and we pick that solution up at $n = 8$

n		$y'_A(0)$	$y'_A(1)$	$y_A(0.05)$	$y_A(0.5)$	x_s
2	S_1	−0.9**693**	−1.30**98**	1.94**85**	1.26**26**	
4	S_1	−1.0067	−1.3078	1.9461	1.2652	
6	S_1	−1.0067	−1.3078	1.9461	1.2652	−2.8341
Num	S_1	−1.0067	−1.3078	1.9461	1.2652	6.5297
8	S_2	34.8**78**	−19.74**2**	3.692**7**	10.17**3**	
10	S_2	34.88**0**	−19.743	3.6928	10.174	
12	S_2	34.881	−19.743	3.6928	10.174	−0.93459
Num	S_2	34.881	−19.743	3.6928	10.174	2.3855

This solution attains the required convergence at $n = 12$. MAPLE only computes the numerical solution for S_1. The numerical solution for S_2 is obtained by iterative shooting from $x = 0$ with $y(0) = 2,\ \ y'(0) = H'_{23}(0)$. The singularities of S_1 and S_2 in the complex plane are shown in the x_s column

To take the limit $\varepsilon \to 0$ we need to modify the two-scale approach slightly to cope with the x-dependence in $a(x, \varepsilon)$. We first make the change of variable

$$z = \frac{1}{\varepsilon}\int_0^x a(t)\mathrm{d}t = \frac{x + x^2/2}{\varepsilon} \tag{4.6}$$

and recast (4.5) with $y(x, \varepsilon) = Y(x, z, \varepsilon)$ so that it becomes

$$\varepsilon^2\frac{\partial^2 Y}{\partial x^2} + \varepsilon(1 + x)\frac{\partial Y}{\partial x} + 3\varepsilon Y + (1 + x)^2\left\{\frac{\partial^2 Y}{\partial z^2} + \frac{\partial Y}{\partial z}\right\} + \varepsilon\frac{\partial Y}{\partial z} + 2\varepsilon(1 + x)\frac{\partial^2 Y}{\partial z\partial x} = 0. \tag{4.7}$$

We now follow the general two-scale procedure except now the right-hand boundary condition is applied, from (4.6), at $x = 1$ and $z = 3/2\varepsilon$. The results of the computations are displayed in Table 4 and compared with the zero-order multiple-scales and the numerical solutions to the problem.

4.3 $b(y, x, \varepsilon)$ has a nonlinear dependence on y

The situation wherein $b(y, x, \varepsilon)$ is a nonlinear function of y introduces a number of new features to the analysis. In this section we will try to outline the salient aspects using the example

$$\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 2, \quad y(1) = 1/2. \tag{4.8}$$

We first observe that this problem does not have a unique solution for $\varepsilon > 0$. This is revealed when we apply the direct method to the problem the results of which are presented for $\varepsilon = 1$ in Table 5.

It is important to say how these two solutions are computed. The two unknowns in the direct method are a_1 and b_1. Since in what follows $x = 0$ attracts either an equal or lower weighting than $x = 1$, we choose to algebraically eliminate a_1 to obtain a single equation for b_1 on which we can perform a real root search. In this way the required five significant figure convergence is achieved for S_1 by $n = 6$ while that for S_2 we have to go to $n = 12$. The numerical solution to the problem is problematical in that the MAPLE software only gives S_1. The solution for S_2 is obtained using iterative shooting from $x = 0$ with $y(0) = 2$ and $H'_{23}(0)$ as a starting value for $y'(0)$. Now that we have estimates for $y'(0)$, we are in a position to search for singularities in the complex plane for S_1 and S_2 to check that they do not affect convergence, by numerically solving the initial-value problem for $y(x)$ for the complex form of the nonlinear equation

$$\varepsilon y'' + y' + y^2 = 0 \tag{4.9}$$

with $y(0) = 2$ and either $y'(0) = H'_{11}(0)$ for S_1 and $y'(0) = H'_{23}(0)$ for S_2. This reveals singularities at $x = x_s = -2.8341$ and 6.5297 for S_1 and $x = x_s = -0.93459$ and 2.3855 for S_2 with the behaviour, derived from (4.9),

$$Y_i \sim -6\varepsilon/(x - x_s)^2$$

which are both well outside the critical ovals for $\mu = 1$ and $\mu = 1/2$. We have not been able to locate any other singularities of the solutions in the complex plane nor are any indicated by convergence difficulties of the direct method.

The two solutions to the problem scale in different ways in the limit $\varepsilon \to 0$. The first solution scales according to

$$x = \varepsilon z, \quad y(x) = Y_1(z),$$

so that

$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d}z^2} + \frac{\mathrm{d}Y_1}{\mathrm{d}z} + \varepsilon Y_1^2 = 0,$$

while the second solution scales like

$$x = \varepsilon z, \quad y(x) = Y_2(z)/\varepsilon,$$

so that

$$\frac{\mathrm{d}^2 Y_2}{\mathrm{d}z^2} + \frac{\mathrm{d}Y_2}{\mathrm{d}z} + Y_2^2 = 0$$

and we have the full equation in the boundary layer. Thus, from the standpoint of boundary-layer theory, only S_1 offers us any tractability, so we concentrate on that solution from now on and continue to reduce ε using the direct method. This presents no new difficulties and the results are presented in the upper half of Table 6 reducing ε to 0.05 by taking $\mu = 1/2$. We note that we may locate the singularities of S_1 for the indicated values of ε in the same way as we did for $\varepsilon = 1$. In each case the singularities do not interfere with the convergence being well outside the respective critical ovals when applying the direct method.

In implementing the two-scale method in the limit $\varepsilon \to 0$ there are a number of new features that we have to take account of. The nonlinearity generates terms involving products of exponentials which it is not necessary to eliminate as secular terms. Thus, the differential equations we have to solve at each stage are inhomogeneous. This is not a difficulty for MAPLE but it means in particular that we cannot write $A_0(z)$ in the form (3.30); hence the boundary condition does not give $\beta_{00}(\varepsilon)$ explicitly in terms of $\alpha_{00}(\varepsilon)$. Thus, we have to solve three nonlinear algebraic equations for the triple $\{\alpha_{00}, \beta_{00}, B_1\}$. The results of these computations are given in the lower half of Table 6.

5 Discussion

In the paper we have shown, using a two-scale technique, how we can obtain uniformly valid polynomial solutions to two-point ordinary boundary-value problems involving a small parameter. A direct method is first applied and, utilising the flexibility offered by fitting different numbers of end point derivatives, we may exploit the convergence theory for Hermite expansions in the complex plane to reduce the value of the small parameter before the direct method eventually fails. This failure can be obviated by using a two-scale method where, again using Hermite expansions, we construct polynomials in the slow variable with coefficients that are functions of the fast variable.

The direct method invites comparison with other methods of constructing polynomial solutions to boundary-value problems such as Chebychev collocation. These methods have the advantage over Hermite-expansion techniques in that they always converge irrespective of the location of singularities of the solution, whereas the Hermite-expansion method will only converge if the singularities lie outside certain prescribed domains. However, it has the distinct advantage that the number of unknowns is fixed, whereas in collocation at selected points the number of unknowns increases with the degree of the expansion. This is particularly important for nonlinear problems

Table 6 Results for the solution S_1 of the nonlinear problem $\varepsilon y'' + y' + y^2 = 0,\quad y(0) = 2,\ y(1) = 1/2$ as ε is reduced

ε	μ	n	$y'_A(0)$	$y'_A(1)$	$y_A(0.05)$	$y_A(0.5)$	x_s	Method
0.5	1	4	−.43**173**	−1.1029	1.969**9**	1.26**27**		Direct
		6	−.4325**6**	−1.1029	1.9698	1.2625		
		8	−.43252	−1.1029	1.9698	1.2625	−1.7449	
							16.151	
		Num	−.43252	−1.1029	1.9698	1.2625		
0.1	1	10	−6.360**8**	−.287**37**	1.7114	.7262**3**		
		12	−6.3610	−.28740	1.7114	.72620		
		14	−6.3610	−.28740	1.7114	.72620	−0.72645	
		Num	−6.3610	−.28740	1.7114	.72620		
		ms0	**−9.0058**	−.2**5145**	1.**6209**	.**68151**		
.05	1/2	8	−16.80**4**	−.26443	1.4121	.68**300**		
		10	−16.805	−.26443	1.4121	.6825**9**		
		12	−16.805	−.26443	1.4121	.6825**0**	−0.40150	
		Num	−16.805	−.26443	1.4121	.68251		
		ms0	−1**9.000**	−.2**5000**	1.4**022**	.6**6667**		
.05		5	−16.80**4**	−.2644**6**	1.4121	.6824**9**		Two-scale
		6	−16.805	−.2644**4**	1.4121	.6824**8**		
		7	−16.805	−.26443	1.4121	.6824**7**		
.01		3	−97.06**5**	−.2525**3**	.972**05**	.669**23**		
		4	−97.05**9**	−.2525**7**	.9721**1**	.6693**3**		
		5	−97.060	−.25256	.97210	.66931	−0.09263	
		Num	−97.060	.25256	.97210	.66932		
		ms0	−9**9.000**	−.25**000**	.9**5981**	.66**667**		
.001		3	−997.1**0**	−.25025	.95356	.6669**1**		
		4	−997.11	−.25025	.95356	.6669**3**		
		5	−997.11	−.25025	.95356	.66692	−0.01146	
		ms0	−99**9.00**	−.250	.95**981**	.666**67**		

Comparison is made where possible with the numerical solution and the zero-order multiple-scales solution to the problem. The singularities of S_1 in the complex plane for various ε are shown in the x_s-column

where in many cases, such as in the nonlinear example of Sect. 4, we may reduce the rootfinding problem to one involving a single variable. In addition, by allowing the number of terms at the two end points to increase at different rates in the convergence process, the Hermite method offers some inbuilt flexibility in allowing us to modify convergence domains and thereby the possibility of maintaining convergence. For linear problems this can often be done in advance since the singularities are determined by the equation. For nonlinear equations this is in general not possible but progress can be made by following a strategy outlined in the nonlinear example of Sect. 4.

In the paper we compare the two-scale method with the zero-order multiple-scales result on the basis that in computing an $H_N(x, z)$ fitting n_0 and n_1 terms at $x = 0$ and $x = 1$, respectively, where $n_0 = n_1 = n$ and $N = 2n + 1$, we neglect terms $O(\varepsilon)$ in $A_n(z, \varepsilon)$. See Eq. (3.20) when $n = 2$. Although it is difficult to be more precise since, as the tabulated results indicate, there is clearly some interplay with n as well as the error involved as $\varepsilon \to 0$, the method seems to work well in the limit.

Our method can be extended to cover more general boundary conditions and also to higher-order equations or systems. We believe that the philosophy of approach could have wide application in boundary-value singular-perturbation problems and will be a useful additional technique for applied mathematicians, engineers and scientists.

Appendix 1: Hermite expansions with remainder

If $f(t)$ is analytic in a simply connected domain D containing a simple closed contour C in the complex t-plane enclosing $t = z$, the Cauchy integral formula states that

$$f(z) = \int_C \frac{f(t)}{(t-z)}\mathrm{d}t. \tag{A.1}$$

We also suppose that C encloses $t = 0$ and $t = 1$. We can write (A.1) as

$$f(z) = \frac{1}{2\pi \mathrm{i}}\int_C \frac{\{w(t)-w(z)\}}{(t-z)w(t)} f(t)\mathrm{d}t + \frac{1}{2\pi \mathrm{i}}\int_C \frac{z^{n_0}(z-1)^{n_1}}{(t-z)t^{n_0}(t-1)^{n_1}} f(t)\mathrm{d}t = P(z) + R(n,z), \tag{A.2}$$

where $w(z) = z^{n_0}(z-1)^{n_1}$. The integrand in $P(z)$ has poles of multiplicity n_0 and n_1 at $t = 0$ and $t = 1$, respectively, within C. Thus we can evaluate $P(z)$ using the residue theorem, which reveals that it is a polynomial of degree $n_0 + n_1 - 1$. Furthermore we can show that

$$P^{(r)}(0) = f^{(r)}(0), \quad r = 0, 1, \ldots.., n_0 - 1$$

and

$$P^{(r)}(1) = f^{(r)}(1), \quad r = 0, 1, \ldots.., n_1 - 1.$$

Thus, by uniqueness of $H_N(z)$

$$P(z) \equiv H_N(z)$$

and we have (2.3). In addition, by bounding $|f(t)|$ and $1/|t-z|$ on C, we obtain

$$|R(n,z)| < K\int_C \left|\frac{z^{n_0}(1-z)^{n_1}}{t^{n_0}(1-t)}\right|\mathrm{d}t = K\int_C \left|\frac{z^{\mu}(1-z)}{t^{\mu}(1-t)}\right|^{nq}\mathrm{d}t,$$

where $n_0 = np, n_1 = nq$ and $\mu = p/q$ for integers p and q. Thus $|R(n,z)| \to 0$ and hence $P(z)$ converges to $f(z)$ as $n \to \infty$ provided $f(z)$ is analytic inside the generalised Cassini oval O_μ for which

$$\left|z^{\mu}(1-z)\right| < r_0, \tag{A.3}$$

where

$$r_0 = \inf_{t\in \mathrm{C}\setminus \mathrm{D}}\left\{\left|t^{\mu}(1-t)\right|\right\}.$$

Appendix 2: Multiple-scales solutions

Here we give the zero-order multiple-scales solutions for the examples in the main body of the paper.

$$\text{I: } \varepsilon y'' + y' - xy = 0, \quad y(0) = y(1) = 1; \quad y(x) = A\mathrm{e}^{x^2/2} + B\mathrm{e}^{-x^2/2}\mathrm{e}^{-z} + O(\varepsilon), \quad 0 \le x \le 1, \tag{A.4}$$

where A and B are obtained by applying the boundary conditions at $x = 0$ and $x = 1$.

$$\text{II: } \varepsilon y'' + (1+x)y' + 3xy = 0, \quad y(0) = y(1) = 1; \tag{A.5}$$

$$y(x) = A\,(1+x)^{-3} + B(1+x)^2 \mathrm{e}^{-(x+x^2/2)/\varepsilon} + O(\varepsilon), \quad 0 \le x \le 1,$$

where A and B are obtained by applying the boundary conditions at $x = 0$ and $x = 1$.

III: $\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 2,\ y(1) = 1/2;$

$$y(x) = A_0(x) + B_0(x)\mathrm{e}^{-x/\varepsilon} + O(\varepsilon), \quad 0 \le x \le 1, \tag{A.6}$$

where

$$A_0(x) = 1/(x + A), \quad B_0(x) = B(x + A)^2.$$

Again A and B are obtained by applying the boundary conditions at $x = 0$ and $x = 1$.

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